0.4 TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

Many phenomena encountered in your daily life involve *waves.* For instance, music is transmitted from radio stations in the form of electromagnetic waves. Your radio receiver decodes these electromagnetic waves and causes a thin membrane inside the speakers to vibrate, which, in turn, creates pressure waves in the air. When these waves reach your ears, you hear the music from your radio (see Figure 0.53). Each of these waves is *periodic,* meaning that the basic shape of the wave is repeated over and over again. The mathematical description of such phenomena involves periodic functions, the most familiar of which are the trigonometric functions. First, we remind you of a basic definition.

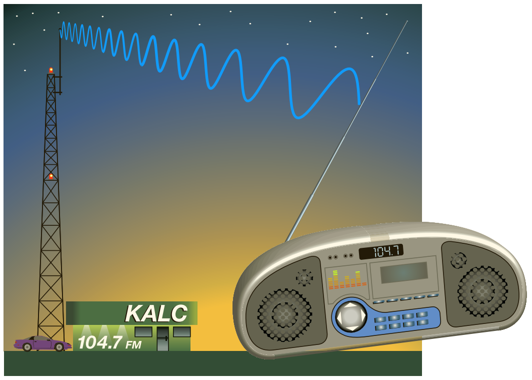


FIGURE 0.53
Radio and sound waves

NOTES

When we discuss the period of a function, we most often focus on the fundamental period.

DEFINITION 4.1

A function f is **periodic** of **period** T if

$$f(x+T) = f(x)$$

for all x such that x and $x+T$ are in the domain of f. The smallest such number $T > 0$ is called the **fundamental period.**

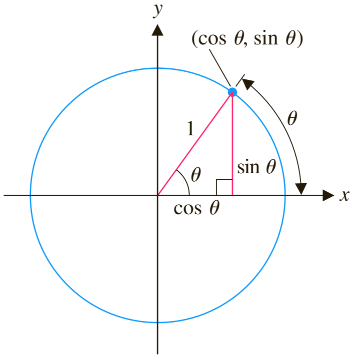


FIGURE 0.54
Definition of $\sin\theta$ and $\cos\theta$:
$\cos\theta = x$ and $\sin\theta = y$

There are several equivalent ways of defining the sine and cosine functions. We want to emphasize a simple definition from which you can easily reproduce many of the basic properties of these functions. Referring to Figure 0.54, begin by drawing the unit circle $x^2 + y^2 = 1$. Let θ be the angle measured (counterclockwise) from the positive x-axis to the line segment connecting the origin to the point (x, y) on the circle. Here, we measure θ in **radians,** where the radian measure of the angle θ is the length of the arc indicated in the figure. Again referring to Figure 0.54, we define $\sin\theta$ to be the y-coordinate of the point on the circle and $\cos\theta$ to be the x-coordinate of the point. From this definition, it follows that $\sin\theta$ and $\cos\theta$ are defined for all values of θ, so that each has domain $-\infty < \theta < \infty$, while the range for each of these functions is the interval $[-1, 1]$.

REMARK 4.1

Unless otherwise noted, we always measure angles in radians.

Note that since the circumference of a circle ($C = 2\pi r$) of radius 1 is 2π, we have that $360°$ corresponds to 2π radians. Similarly, $180°$ corresponds to π radians, $90°$ corresponds to $\pi/2$ radians, and so on. In the accompanying table, we list some common angles as measured in degrees, together with the corresponding radian measures.

Angle in degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$135°$	$180°$	$270°$	$360°$
Angle in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

THEOREM 4.1

The functions $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ are periodic, of period 2π.

PROOF

Referring to Figure 0.54, since a complete circle is 2π radians, adding 2π to any angle takes you all the way around the circle and back to the same point (x, y). This says that

$$\sin(\theta + 2\pi) = \sin\theta$$

and

$$\cos(\theta + 2\pi) = \cos\theta,$$

for all values of θ. Furthermore, 2π is the smallest angle for which this is true. ■

You are likely already familiar with the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ shown in Figures 0.55a and 0.55b, respectively.

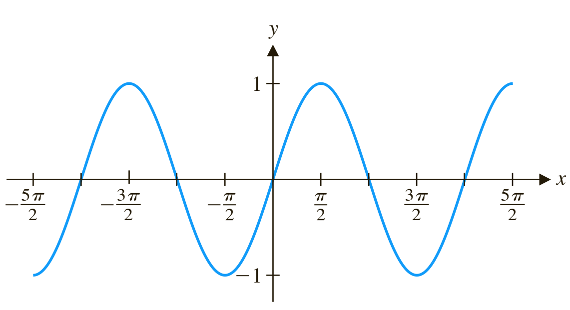


FIGURE 0.55a
$y = \sin x$

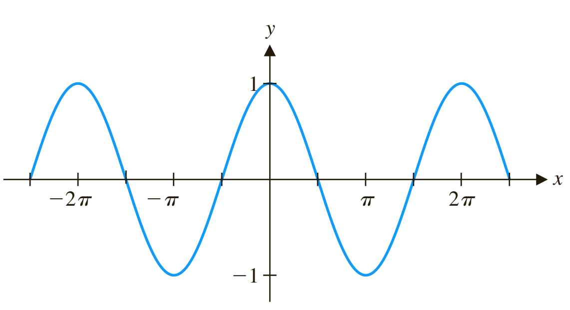


FIGURE 0.55b
$y = \cos x$

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1

Notice that you could slide the graph of $y = \sin x$ slightly to the left or right and get an exact copy of the graph of $y = \cos x$. Specifically, we have the relationship

$$\boxed{\sin\left(x + \frac{\pi}{2}\right) = \cos x.}$$

The accompanying table lists some common values of sine and cosine. Notice that many of these can be read directly from Figure 0.54.

EXAMPLE 4.1 Solving Equations Involving Sines and Cosines

Find all solutions of the equations (a) $2\sin x - 1 = 0$ and (b) $\cos^2 x - 3\cos x + 2 = 0$.

Solution For (a), notice that $2\sin x - 1 = 0$ if $2\sin x = 1$ or $\sin x = \frac{1}{2}$. From the unit circle, we find that $\sin x = \frac{1}{2}$ if $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Since $\sin x$ has period 2π, additional solutions are $\frac{\pi}{6} + 2\pi$, $\frac{5\pi}{6} + 2\pi$, $\frac{\pi}{6} + 4\pi$ and so on. A convenient way of indicating that *any* integer multiple of 2π can be added to either solution is to write $x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$, for any integer n. Part (b) may look rather difficult at first. However, notice that it looks like a quadratic equation using $\cos x$ instead of x. With this clue, you can factor the left-hand side to get

$$0 = \cos^2 x - 3\cos x + 2 = (\cos x - 1)(\cos x - 2),$$

from which it follows that either $\cos x = 1$ or $\cos x = 2$. Since $-1 \le \cos x \le 1$ for all x, the equation $\cos x = 2$ has no solution. However, we get $\cos x = 1$ if $x = 0, 2\pi$ or any integer multiple of 2π. We can summarize all the solutions by writing $x = 2n\pi$, for any integer n. ■

REMARK 4.2

Instead of writing $(\sin\theta)^2$ or $(\cos\theta)^2$, we usually use the notation $\sin^2\theta$ and $\cos^2\theta$, respectively.

We now give definitions of the remaining four trigonometric functions.

DEFINITION 4.2

The **tangent** function is defined by $\tan x = \dfrac{\sin x}{\cos x}$.

The **cotangent** function is defined by $\cot x = \dfrac{\cos x}{\sin x}$.

The **secant** function is defined by $\sec x = \dfrac{1}{\cos x}$.

The **cosecant** function is defined by $\csc x = \dfrac{1}{\sin x}$.

REMARK 4.3

Most calculators have keys for the functions $\sin x$, $\cos x$ and $\tan x$, but not for the other three trigonometric functions. This reflects the central role that $\sin x$, $\cos x$ and $\tan x$ play in applications. To calculate function values for the other three trigonometric functions, you can simply use the identities

$$\cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}$$

$$\text{and} \quad \csc x = \frac{1}{\sin x}.$$

We show graphs of these functions in Figures 0.56a, 0.56b, 0.56c and 0.56d. Notice in each graph the locations of the vertical asymptotes. For the "co" functions $\cot x$ and $\csc x$, the division by $\sin x$ causes vertical asymptotes at $0, \pm\pi, \pm 2\pi$ and so on (where $\sin x = 0$). For $\tan x$ and $\sec x$, the division by $\cos x$ produces vertical asymptotes at $\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$ and so on (where $\cos x = 0$). Once you have determined the vertical asymptotes, the graphs are relatively easy to draw.

Notice that $\tan x$ and $\cot x$ are periodic, of period π, while $\sec x$ and $\csc x$ are periodic, of period 2π.

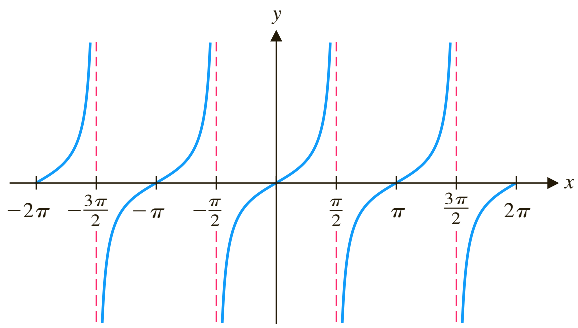


FIGURE 0.56a
$y = \tan x$

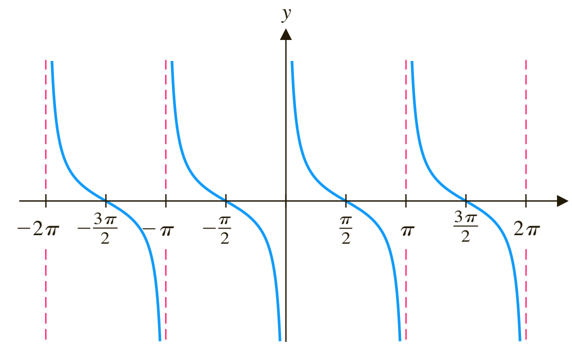


FIGURE 0.56b
$y = \cot x$

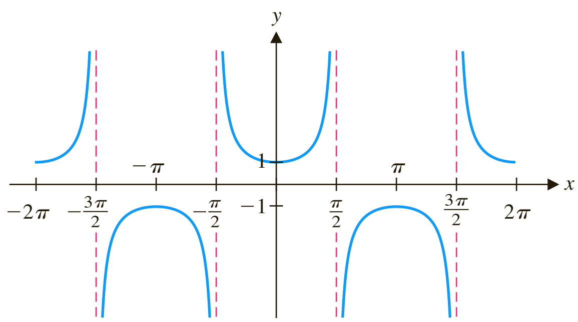


FIGURE 0.56c
$y = \sec x$

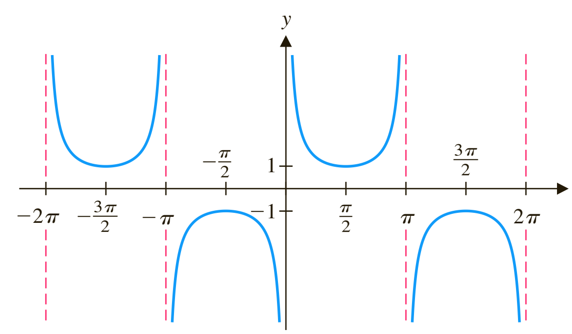


FIGURE 0.56d
$y = \csc x$

It is important to learn the effect of slight modifications of these functions. We present a few ideas here and in the exercises.

EXAMPLE 4.2 Altering Amplitude and Period

Graph $y = 2\sin x$ and $y = \sin 2x$, and describe how each differs from the graph of $y = \sin x$ (see Figure 0.57a).

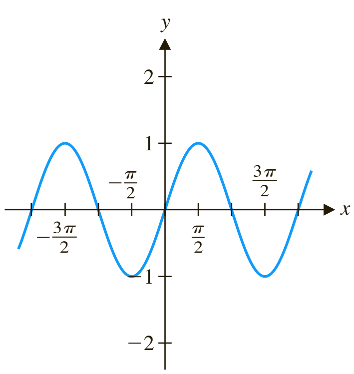


FIGURE 0.57a
$y = \sin x$

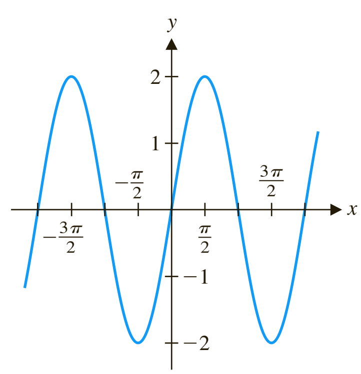


FIGURE 0.57b
$y = 2\sin x$

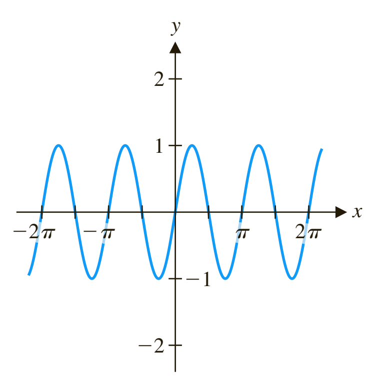


FIGURE 0.57c
$y = \sin(2x)$

Solution The graph of $y = 2 \sin x$ is given in Figure 0.57b. Notice that this graph is similar to the graph of $y = \sin x$, except that the y-values oscillate between -2 and 2 instead of -1 and 1. Next, the graph of $y = \sin 2x$ is given in Figure 0.57c. In this case, the graph is similar to the graph of $y = \sin x$ except that the period is π instead of 2π (so that the oscillations occur twice as fast). ■

The results in example 4.2 can be generalized. For $A > 0$, the graph of $y = A \sin x$ oscillates between $y = -A$ and $y = A$. In this case, we call A the **amplitude** of the sine curve. Notice that for any positive constant c, the period of $y = \sin cx$ is $2\pi/c$. Similarly, for the function $A \cos cx$, the amplitude is A and the period is $2\pi/c$.

The sine and cosine functions can be used to model sound waves. A pure tone (think of a single flute note) is a pressure wave described by the sinusoidal function $A \sin ct$. (Here, we are using the variable t, since the air pressure is a function of *time*.) The amplitude A determines how loud the tone is perceived to be and the period determines the pitch of the note. In this setting, it is convenient to talk about the **frequency** $f = c/2\pi$. The higher the frequency is, the higher the pitch of the note will be. (Frequency is measured in hertz, where 1 hertz equals 1 cycle per second.) Note that the frequency is simply the reciprocal of the period.

EXAMPLE 4.3 Finding Amplitude, Period and Frequency

Find the amplitude, period and frequency of (a) $f(x) = 4 \cos 3x$ and (b) $g(x) = 2 \sin(x/3)$.

Solution (a) For $f(x)$, the amplitude is 4, the period is $2\pi/3$ and the frequency is $3/(2\pi)$ (see Figure 0.58a). (b) For $g(x)$, the amplitude is 2, the period is $2\pi/(1/3) = 6\pi$ and the frequency is $1/(6\pi)$ (see Figure 0.58b).

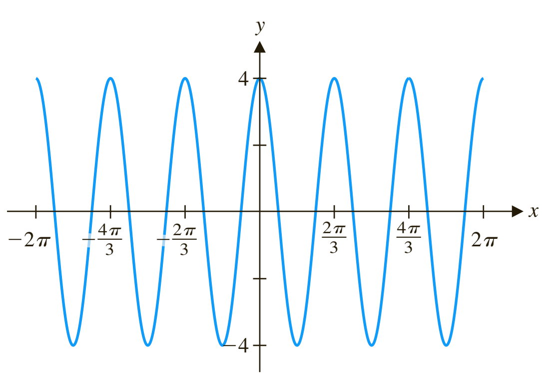


FIGURE 0.58a
$y = 4 \cos 3x$

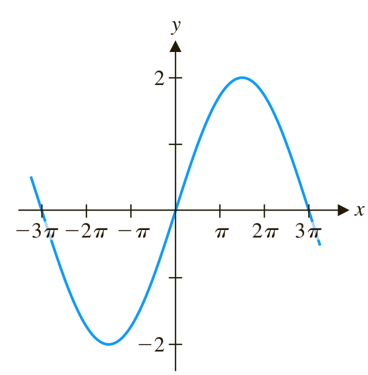


FIGURE 0.58b
$y = 2 \sin(x/3)$

■

There are numerous formulas or **identities** that are helpful in manipulating the trigonometric functions. You should observe that, from the definition of $\sin \theta$ and $\cos \theta$ (see Figure 0.54), the Pythagorean Theorem gives us the familiar identity

$$\sin^2 \theta + \cos^2 \theta = 1,$$

since the hypotenuse of the indicated triangle is 1. This is true for any angle θ. In addition,

$$\sin(-\theta) = -\sin \theta \quad \text{and} \quad \cos(-\theta) = \cos \theta$$

We list several important identities in Theorem 4.2.

THEOREM 4.2

For any real numbers α and β, the following identities hold:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \tag{4.1}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{4.2}$$
$$\sin^2\alpha = \tfrac{1}{2}(1-\cos 2\alpha) \tag{4.3}$$
$$\cos^2\alpha = \tfrac{1}{2}(1+\cos 2\alpha). \tag{4.4}$$

From the basic identities summarized in Theorem 4.2, numerous other useful identities can be derived. We derive two of these in example 4.4.

EXAMPLE 4.4 Deriving New Trigonometric Identities

Derive the identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Solution These can be obtained from formulas (4.1) and (4.2), respectively, by substituting $\alpha = \theta$ and $\beta = \theta$. Alternatively, the identity for $\cos 2\theta$ can be obtained by subtracting equation (4.3) from equation (4.4). ■

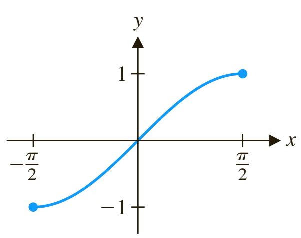


FIGURE 0.59
$y = \sin x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The Inverse Trigonometric Functions

We now expand the set of functions available to you by defining inverses to the trigonometric functions. To get started, look at a graph of $y = \sin x$ (see Figure 0.57a). Notice that we cannot define an inverse function, since $\sin x$ is not one-to-one. Although the sine function does not have an inverse function, we can define one by modifying the domain of the sine. We do this by choosing a portion of the sine curve that passes the horizontal line test. If we restrict the domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $y = \sin x$ is one-to-one there (see Figure 0.59) and, hence, has an inverse. We thus define the **inverse sine** function by

$$\boxed{y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x \textit{ and } -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2}.} \tag{4.5}$$

Think of this definition as follows: if $y = \sin^{-1} x$, then y is the angle $\left(\text{between } -\frac{\pi}{2} \text{ and } \frac{\pi}{2}\right)$ for which $\sin y = x$. Note that we could have selected any interval on which $\sin x$ is one-to-one, but $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the most convenient. To verify that these are inverse functions, observe that

$$\sin(\sin^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$$

and

$$\sin^{-1}(\sin x) = x, \quad \text{for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{4.6}$$

Read equation (4.6) very carefully. It *does not* say that $\sin^{-1}(\sin x) = x$ for *all* x, but rather, *only* for those in the restricted domain, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For instance, $\sin^{-1}(\sin\pi) \ne \pi$, since

$$\sin^{-1}(\sin\pi) = \sin^{-1}(0) = 0.$$

REMARK 4.4

Mathematicians often use the notation **arcsin *x*** in place of $\sin^{-1} x$. People read $\sin^{-1} x$ interchangeably as "inverse sine of x" or "arcsine of x."

EXAMPLE 4.5 Evaluating the Inverse Sine Function

Evaluate (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution For (a), we look for the angle θ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin\theta = \frac{\sqrt{3}}{2}$. Note that since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have that $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. For (b), note that $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus,

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

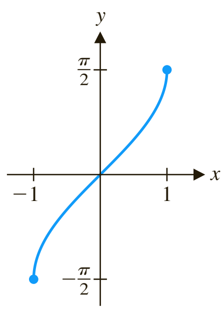


FIGURE 0.60
$y = \sin^{-1} x$

Judging by example 4.5, you might think that (4.5) is a roundabout way of defining a function. If so, you've got the idea exactly. In fact, we want to emphasize that what we know about the inverse sine function is principally through reference to the sine function.

Recall from our discussion in section 0.3 that we can draw a graph of $y = \sin^{-1} x$ simply by reflecting the graph of $y = \sin x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (from Figure 0.59) through the line $y = x$ (see Figure 0.60).

Turning to $y = \cos x$, observe that restricting the domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, as we did for the inverse sine function, will not work here. (Why not?) The simplest way to make $\cos x$ one-to-one is to restrict its domain to the interval $[0, \pi]$ (see Figure 0.61). Consequently, we define the **inverse cosine** function by

$$\boxed{y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x \textit{ and } 0 \le y \le \pi.}$$

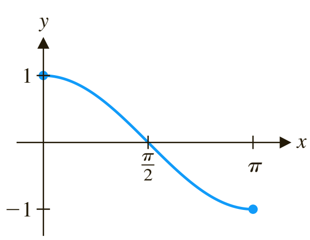


FIGURE 0.61
$y = \cos x$ on $[0, \pi]$

Note that here, we have

$$\cos(\cos^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$$

and
$$\cos^{-1}(\cos x) = x, \quad \text{for all } x \in [0, \pi].$$

As with the definition of arcsine, it is helpful to think of $\cos^{-1} x$ as that angle θ in $[0, \pi]$ for which $\cos\theta = x$. As with $\sin^{-1} x$, it is common to use $\cos^{-1} x$ and $\arccos x$ interchangeably.

EXAMPLE 4.6 Evaluating the Inverse Cosine Function

Evaluate (a) $\cos^{-1}(0)$ and (b) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Solution For (a), you will need to find that angle θ in $[0, \pi]$ for which $\cos\theta = 0$. It's not hard to see that $\cos^{-1}(0) = \frac{\pi}{2}$. (If you calculate this on your calculator and get 90, your calculator is in degrees mode. In this event, you should immediately change it to radians mode.) For (b), look for the angle $\theta \in [0, \pi]$ for which $\cos\theta = -\frac{\sqrt{2}}{2}$. Notice that $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\frac{3\pi}{4} \in [0, \pi]$. Consequently,

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}.$$

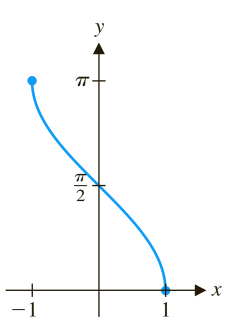


FIGURE 0.62
$y = \cos^{-1} x$

Once again, we obtain the graph of this inverse function by reflecting the graph of $y = \cos x$ on the interval $[0, \pi]$ (seen in Figure 0.61) through the line $y = x$ (see Figure 0.62).

We can define inverses for each of the four remaining trigonometric functions in similar ways. For $y = \tan x$, we restrict the domain to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Think about why the endpoints of this interval are not included (see Figure 0.63). Having done this, you should readily see that we define the **inverse tangent** function by

$$\boxed{y = \tan^{-1} x \quad \text{if and only if} \quad \tan y = x \ \textit{and} \ -\tfrac{\pi}{2} < y < \tfrac{\pi}{2}.}$$

The graph of $y = \tan^{-1} x$ is then as seen in Figure 0.64, found by reflecting the graph in Figure 0.63 through the line $y = x$.

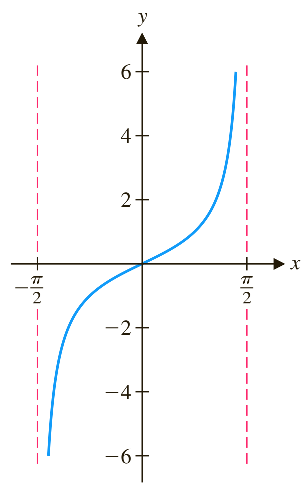


FIGURE 0.63
$y = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

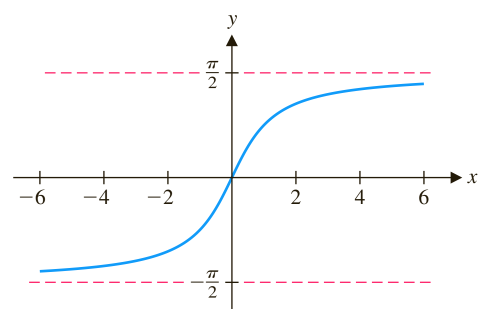


FIGURE 0.64
$y = \tan^{-1} x$

EXAMPLE 4.7 Evaluating an Inverse Tangent

Evaluate $\tan^{-1}(1)$.

Solution You must look for the angle θ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $\tan\theta = 1$. This is easy enough. Since $\tan\left(\frac{\pi}{4}\right) = 1$ and $\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have that $\tan^{-1}(1) = \frac{\pi}{4}$. ■

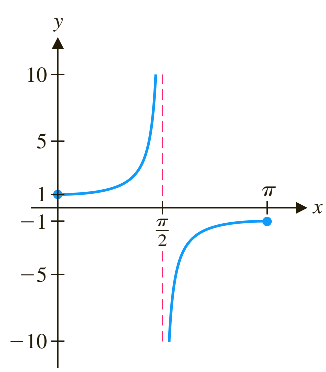


FIGURE 0.65
$y = \sec x$ on $[0, \pi]$

We now turn to defining an inverse for $\sec x$. First, we must issue a disclaimer. There are several reasonable ways in which to suitably restrict the domain and different authors restrict it differently. We have (somewhat arbitrarily) chosen to restrict the domain to be $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Why not use all of $[0, \pi]$? You need only think about the definition of $\sec x$ to see why we needed to exclude the value $x = \frac{\pi}{2}$. See Figure 0.65 for a graph of $\sec x$ on this domain. (Note the vertical asymptote at $x = \frac{\pi}{2}$.) Consequently, we define the **inverse secant** function by

$$\boxed{y = \sec^{-1} x \quad \text{if and only if} \quad \sec y = x \ \textit{and} \ y \in \left[0, \tfrac{\pi}{2}\right) \cup \left(\tfrac{\pi}{2}, \pi\right].}$$

A graph of $\sec^{-1} x$ is shown in Figure 0.66.

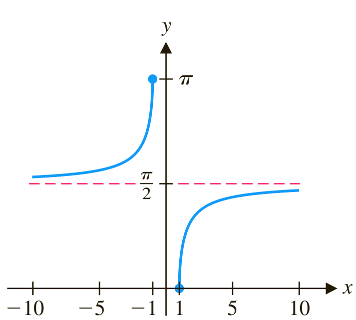


FIGURE 0.66
$y = \sec^{-1} x$

EXAMPLE 4.8 Evaluating an Inverse Secant

Evaluate $\sec^{-1}(-\sqrt{2})$.

REMARK 4.5

We can likewise define inverses to $\cot x$ and $\csc x$. As these functions are used only infrequently, we will omit them here and examine them in the exercises.

Function	*Domain*	*Range*
$\sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

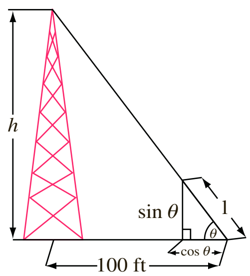


FIGURE 0.67
Height of a tower

Solution You must look for the angle θ with $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, for which $\sec\theta = -\sqrt{2}$. Notice that if $\sec\theta = -\sqrt{2}$, then $\cos\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$. Recall from example 4.6 that $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$. Further, the angle $\frac{3\pi}{4}$ is in the interval $\left(\frac{\pi}{2}, \pi\right]$ and so, $\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}$. ■

Calculators do not usually have built-in functions for $\sec x$ or $\sec^{-1} x$. In this case, you must convert the desired secant value to a cosine value and use the inverse cosine function, as we did in example 4.8.

We summarize the domains and ranges of the three main inverse trigonometric functions in the margin.

In many applications, we need to calculate the length of one side of a right triangle using the length of another side and an **acute** angle (i.e., an angle between 0 and $\frac{\pi}{2}$ radians). We can do this rather easily, as in example 4.9.

EXAMPLE 4.9 Finding the Height of a Tower

A person 100 feet from the base of a tower measures an angle of 60° from the ground to the top of the tower (see Figure 0.67). (a) Find the height of the tower. (b) What angle is measured if the person is 200 feet from the base?

Solution For (a), we first convert 60° to radians:

$$60^\circ = 60\frac{\pi}{180} = \frac{\pi}{3} \text{ radians.}$$

We are given that the base of the triangle in Figure 0.67 is 100 feet. We must now compute the height h of the tower. Using the similar triangles indicated in Figure 0.67, we have

$$\frac{\sin\theta}{\cos\theta} = \frac{h}{100},$$

so that the height of the tower is

$$h = 100\frac{\sin\theta}{\cos\theta} = 100\tan\theta = 100\tan\frac{\pi}{3} = 100\sqrt{3} \approx 173 \text{ feet.}$$

For part (b), the similar triangles in Figure 0.67 give us

$$\tan\theta = \frac{h}{200} = \frac{100\sqrt{3}}{200} = \frac{\sqrt{3}}{2}.$$

Since $0 < \theta < \dfrac{\pi}{2}$, we have

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) \approx 0.7137 \text{ radians (about 41 degrees).}$$ ■

In example 4.10, we simplify expressions involving both trigonometric and inverse trigonometric functions.

EXAMPLE 4.10 Simplifying Expressions Involving Inverse Trigonometric Functions

Simplify (a) $\sin(\cos^{-1} x)$ and (b) $\tan(\cos^{-1} x)$.

Solution Do not look for some arcane formula to help you out. *Think* first: $\cos^{-1} x$ is the angle (call it θ) for which $x = \cos\theta$. First, consider the case where $x > 0$. Looking

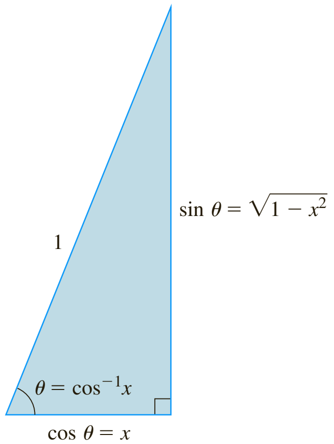


FIGURE 0.68
$\theta = \cos^{-1} x$

at Figure 0.68, we have drawn a right triangle, with hypotenuse 1 and adjacent angle θ. From the definition of the sine and cosine, then, we have that the base of the triangle is $\cos\theta = x$ and the altitude is $\sin\theta$, which by the Pythagorean Theorem is

$$\sin(\cos^{-1} x) = \sin\theta = \sqrt{1-x^2}.$$

Wait! We have not yet finished part (a). Figure 0.68 shows $0 < \theta < \frac{\pi}{2}$, but by definition, $\theta = \cos^{-1} x$ could range from 0 to π. Does our answer change if $\frac{\pi}{2} < \theta < \pi$? To see that it doesn't change, note that if $0 \le \theta \le \pi$, then $\sin\theta \ge 0$. From the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we get

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-x^2}.$$

Since $\sin\theta \ge 0$, we must have

$$\sin\theta = \sqrt{1-x^2},$$

for *all* values of x.

For part (b), you can read from Figure 0.68 that

$$\tan(\cos^{-1} x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}.$$

Note that this last identity is valid, regardless of whether $x = \cos\theta$ is positive or negative. ■

EXERCISES 0.4

WRITING EXERCISES

1. Many students are comfortable using degrees to measure angles and don't understand why they must learn radian measures. As discussed in the text, radians directly measure distance along the unit circle. Distance is an important aspect of many applications. In addition, we will see later that many calculus formulas are simpler in radians form than in degrees. Aside from familiarity, discuss any and all advantages of degrees over radians. On balance, which is better?

2. A student graphs $f(x) = \cos x$ on a graphing calculator and gets what appears to be a straight line at height $y = 1$ instead of the usual cosine curve. Upon investigation, you discover that the calculator has graphing window $-10 \le x \le 10$, $-10 \le y \le 10$ and is in degrees mode. Explain what went wrong and how to correct it.

3. Inverse functions are necessary for solving equations. The restricted range we had to use to define inverses of the trigonometric functions also restricts their usefulness in equation solving. Explain how to use $\sin^{-1} x$ to find all solutions of the equation $\sin u = x$.

4. Discuss how to compute $\sec^{-1} x$, $\csc^{-1} x$ and $\cot^{-1} x$ on a calculator that has built-in functions only for $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$.

5. In example 4.3, $f(x) = 4\cos 3x$ has period $2\pi/3$ and $g(x) = 2\sin(x/3)$ has period 6π. Explain why the sum $h(x) = 4\cos 3x + 2\sin(x/3)$ has period 6π.

6. Give a different range for $\sec^{-1} x$ than that given in the text. For which x's would the value of $\sec^{-1} x$ change? Using the calculator discussion in exercise 4, give one reason why we might have chosen the range that we did.

In exercises 1 and 2, convert the given radians measure to degrees.

1. (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{4\pi}{3}$
2. (a) $\frac{3\pi}{5}$ (b) $\frac{\pi}{7}$ (c) 2 (d) 3

In exercises 3 and 4, convert the given degrees measure to radians.

3. (a) 180° (b) 270° (c) 120° (d) 30°
4. (a) 40° (b) 80° (c) 450° (d) 390°

In exercises 5–14, find all solutions of the given equation.

5. $2\cos x - 1 = 0$
6. $2\sin x + 1 = 0$
7. $\sqrt{2}\cos x - 1 = 0$
8. $2\sin x - \sqrt{3} = 0$

9. $\sin^2 x - 4\sin x + 3 = 0$

10. $\sin^2 x - 2\sin x - 3 = 0$

11. $\sin^2 x + \cos x - 1 = 0$

12. $\sin 2x - \cos x = 0$

13. $\cos^2 x + \cos x = 0$

14. $\sin^2 x - \sin x = 0$

In exercises 15–24, sketch a graph of the function.

15. $f(x) = \sin 2x$

16. $f(x) = \cos 3x$

17. $f(x) = \tan 2x$

18. $f(x) = \sec 3x$

19. $f(x) = 3\cos(x - \pi/2)$

20. $f(x) = 4\cos(x + \pi)$

21. $f(x) = \sin 2x - 2\cos 2x$

22. $f(x) = \cos 3x - \sin 3x$

23. $f(x) = \sin x \sin 12x$

24. $f(x) = \sin x \cos 12x$

In exercises 25–32, identify the amplitude, period and frequency.

25. $f(x) = 3\sin 2x$

26. $f(x) = 2\cos 3x$

27. $f(x) = 5\cos 3x$

28. $f(x) = 3\sin 5x$

29. $f(x) = 3\cos(2x - \pi/2)$

30. $f(x) = 4\sin(3x + \pi)$

31. $f(x) = -4\sin x$

32. $f(x) = -2\cos 3x$

In exercises 33–36, prove that the given trigonometric identity is true.

33. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

34. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

35. (a) $\cos(2\theta) = 2\cos^2\theta - 1$ (b) $\cos(2\theta) = 1 - 2\sin^2\theta$

36. (a) $\sec^2\theta = \tan^2\theta + 1$ (b) $\csc^2\theta = \cot^2\theta + 1$

In exercises 37–46, evaluate the inverse function by sketching a unit circle and locating the correct angle on the circle.

37. $\cos^{-1} 0$

38. $\tan^{-1} 0$

39. $\sin^{-1}(-1)$

40. $\cos^{-1}(1)$

41. $\sec^{-1} 1$

42. $\tan^{-1}(-1)$

43. $\sec^{-1} 2$

44. $\csc^{-1} 2$

45. $\cot^{-1} 1$

46. $\tan^{-1}\sqrt{3}$

47. Prove that, for some constant β,

$$4\cos x - 3\sin x = 5\cos(x + \beta).$$

Then, estimate the value of β.

48. Prove that, for some constant β,

$$2\sin x + \cos x = \sqrt{5}\sin(x + \beta).$$

Then, estimate the value of β.

In exercises 49–52, determine whether the function is periodic. If it is periodic, find the smallest (fundamental) period.

49. $f(x) = \cos 2x + 3\sin \pi x$

50. $f(x) = \sin x - \cos\sqrt{2}x$

51. $f(x) = \sin 2x - \cos 5x$

52. $f(x) = \cos 3x - \sin 7x$

In exercises 53–56, use the range for θ to determine the indicated function value.

53. $\sin\theta = \frac{1}{3}, 0 \le \theta \le \frac{\pi}{2}$; find $\cos\theta$.

54. $\cos\theta = \frac{4}{5}, 0 \le \theta \le \frac{\pi}{2}$; find $\sin\theta$.

55. $\sin\theta = \frac{1}{2}, \frac{\pi}{2} \le \theta \le \pi$; find $\cos\theta$.

56. $\sin\theta = \frac{1}{2}, \frac{\pi}{2} \le \theta \le \pi$; find $\tan\theta$.

In exercises 57–64, use a triangle to simplify each expression. Where applicable, state the range of x's for which the simplification holds.

57. $\cos(\sin^{-1} x)$

58. $\cos(\tan^{-1} x)$

59. $\tan(\sec^{-1} x)$

60. $\cot(\cos^{-1} x)$

61. $\sin\left(\cos^{-1}\frac{1}{2}\right)$

62. $\cos\left(\sin^{-1}\frac{1}{2}\right)$

63. $\tan\left(\cos^{-1}\frac{3}{5}\right)$

64. $\csc\left(\sin^{-1}\frac{2}{3}\right)$

In exercises 65–68, use a graphing calculator or computer to determine the number of solutions of each equation, and numerically estimate the solutions (x is in radians).

65. $2\cos x = 2 - x$

66. $3\sin x = x$

67. $\cos x = x^2 - 2$

68. $\sin x = x^2$

69. A person sitting 2 miles from a rocket launch site measures 20° up to the current location of the rocket. How high up is the rocket?

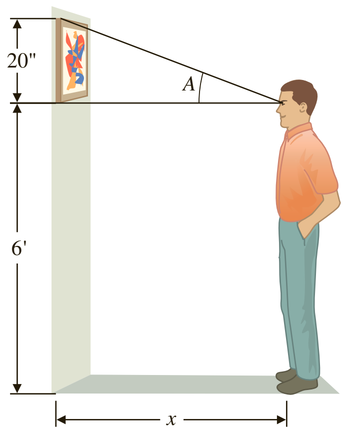
70. A person who is 6 feet tall stands 4 feet from the base of a light pole and casts a 2-foot-long shadow. How tall is the light pole?

71. A surveyor stands 80 feet from the base of a building and measures an angle of 50° to the top of the steeple on top of the building. The surveyor figures that the center of the steeple lies 20 feet inside the front of the structure. Find the distance from the ground to the top of the steeple.

72. Suppose that the surveyor of exercise 71 estimates that the center of the steeple lies between $20'$ and $21'$ inside the front of the structure. Determine how much the extra foot would change the calculation of the height of the building.

73. A picture hanging in an art gallery has a frame 20 inches high, and the bottom of the frame is 6 feet above the floor. A person whose eyes are 6 feet above the floor stands x feet from the wall. Let A be the angle formed by the ray from the person's eye to the bottom of the frame and the ray from the person's

eye to the top of the frame. Write A as a function of x and graph $y = A(x)$.



74. In golf, the goal is to hit a ball into a hole of diameter 4.5 inches. Suppose a golfer stands x feet from the hole trying to putt the ball into the hole. A first approximation of the margin of error in a putt is to measure the angle A formed by the ray from the ball to the right edge of the hole and the ray from the ball to the left edge of the hole. Find A as a function of x.

75. In an AC circuit, the voltage is given by $v(t) = v_p \sin 2\pi f t$, where v_p is the peak voltage and f is the frequency in Hz. A voltmeter actually measures an average (called the **root-mean-square**) voltage, equal to $v_p/\sqrt{2}$. If the voltage has amplitude 170 and period $\pi/30$, find the frequency and meter voltage.

76. An old-style LP record player rotates records at $33\frac{1}{3}$ rpm (revolutions per minute). What is the period (in minutes) of the rotation? What is the period for a 45-rpm record?

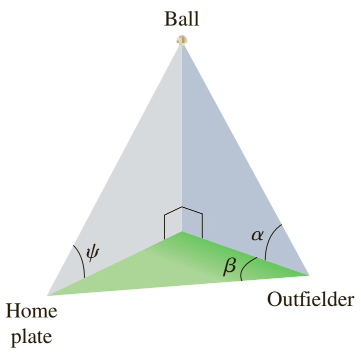
77. Suppose that the ticket sales of an airline (in thousands of dollars) is given by $s(t) = 110 + 2t + 15 \sin\left(\frac{1}{6}\pi t\right)$, where t is measured in months. What real-world phenomenon might cause the fluctuation in ticket sales modeled by the sine term? Based on your answer, what month corresponds to $t = 0$? Disregarding seasonal fluctuations, by what amount is the airline's sales increasing annually?

78. Piano tuners sometimes start by striking a tuning fork and then the corresponding piano key. If the tuning fork and piano note each have frequency 8, then the resulting sound is $\sin 8t + \sin 8t$. Graph this. If the piano is slightly out-of-tune at frequency 8.1, the resulting sound is $\sin 8t + \sin 8.1t$. Graph this and explain how the piano tuner can hear the small difference in frequency.

79. Give precise definitions of $\csc^{-1} x$ and $\cot^{-1} x$.

80. In baseball, outfielders are able to easily track down and catch fly balls that have very long and high trajectories. Physicists have argued for years about how this is done. A recent explanation involves the following geometry.

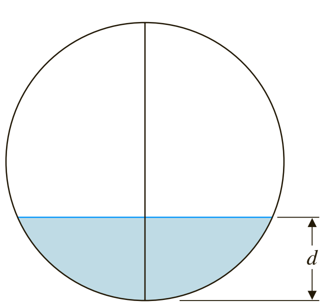
The player can catch the ball by running to keep the angle ψ constant (this makes it appear that the ball is moving in a straight line). Assuming that all triangles shown are right triangles, show that $\tan \psi = \dfrac{\tan \alpha}{\tan \beta}$ and then solve for ψ.



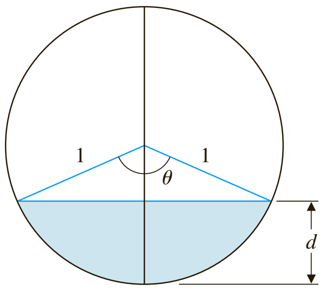
EXPLORATORY EXERCISES

1. In his book and video series *The Ring of Truth,* physicist Philip Morrison performed an experiment to estimate the circumference of the earth. In Nebraska, he measured the angle to a bright star in the sky, then drove 370 miles due south into Kansas and measured the new angle to the star. Some geometry shows that the difference in angles, about 5.02°, equals the angle from the center of the earth to the two locations in Nebraska and Kansas. If the earth is perfectly spherical (it's not) and the circumference of the portion of the circle measured out by 5.02° is 370 miles, estimate the circumference of the earth. This experiment was based on a similar experiment by the ancient Greek scientist Eratosthenes. The ancient Greeks and the Spaniards of Columbus' day knew that the earth was round, they just disagreed about the circumference. Columbus argued for a figure about half of the actual value, since a ship couldn't survive on the water long enough to navigate the true distance.

2. An oil tank with circular cross sections lies on its side. A stick is inserted in a hole at the top and used to measure the depth d of oil in the tank. Based on this measurement, the goal is to compute the percentage of oil left in the tank.



To simplify calculations, suppose the circle is a unit circle with center at (0, 0). Sketch radii extending from the origin to the top of the oil. The area of oil at the bottom equals the area of the portion of the circle bounded by the radii minus the area of the triangle formed above the oil in the figure.



Start with the triangle, which has area one-half base times height. Explain why the height is $1-d$. Find a right triangle in the figure (there are two of them) with hypotenuse 1 (the radius of the circle) and one vertical side of length $1-d$. The horizontal side has length equal to one-half the base of the larger triangle. Show that this equals $\sqrt{1-(1-d)^2}$. The area of the portion of the circle equals $\pi\theta/2\pi = \theta/2$, where θ is the angle at the top of the triangle. Find this angle as a function of d. (Hint: Go back to the right triangle used above with upper angle $\theta/2$.) Then find the area filled with oil and divide by π to get the portion of the tank filled with oil.

3. Computer graphics can be misleading. This exercise works best using a "disconnected" graph (individual dots, not connected). Graph $y = \sin x^2$ using a graphing window for which each pixel represents a step of 0.1 in the x- or y-direction. You should get the impression of a sine wave that oscillates more and more rapidly as you move to the left and right. Next, change the graphing window so that the middle of the original screen (probably $x = 0$) is at the far left of the new screen. You will likely see what appears to be a random jumble of dots. Continue to change the graphing window by increasing the x-values. Describe the patterns or lack of patterns that you see. You should find one pattern that looks like two rows of dots across the top and bottom of the screen; another pattern looks like the original sine wave. For each pattern that you find, pick adjacent points with x-coordinates a and b. Then change the graphing window so that $a \le x \le b$ and find the portion of the graph that is missing. Remember that, whether the points are connected or not, computer graphs always leave out part of the graph; it is part of your job to know whether or not the missing part is important.

0.5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Some bacteria reproduce very quickly, as you may have discovered if you have ever had an infected cut or strep throat. Under the right circumstances, the number of bacteria in certain cultures will double in as little as an hour. In this section, we discuss some functions that can be used to model such rapid growth.

Suppose that initially there are 100 bacteria at a given site and the population doubles every hour. Call the population function $P(t)$, where t represents time (in hours) and start the clock running at time $t = 0$. Since the initial population is 100, we have $P(0) = 100$. After 1 hour, the population will double to 200, so that $P(1) = 200$. After another hour, the population will have doubled again to 400, making $P(2) = 400$ and so on.

To compute the bacterial population after 10 hours, you could calculate the population at 4 hours, 5 hours and so on, or you could use the following shortcut. To find $P(1)$, double the initial population, so that $P(1) = 2 \cdot 100$. To find $P(2)$, double the population at time $t = 1$, so that $P(2) = 2 \cdot 2 \cdot 100 = 2^2 \cdot 100$. Similarly, $P(3) = 2^3 \cdot 100$. This pattern leads us to

$$P(10) = 2^{10} \cdot 100 = 102{,}400.$$

Observe that the population can be modeled by the function

$$P(t) = 2^t \cdot 100.$$

We call $P(t)$ an **exponential** function because the variable t is in the exponent. There is a subtle question here: what is the domain of this function? We have so far used only integer